

A Note on CDE function

Shiro Takeda (Kyoto Sangyo University)

December 5, 2012

Abstract

A small note on CDE function (constant difference of elasticities function, Hertel, Horridge and Pearson 1992) used in the GTAP standard model.¹

Case 1

An expression used in Hertel et al. (1992) Hertel (1997) McDougall (2003).

Expenditure function: Expenditure function for CDE utility function is defined implicitly as

$$\sum_i \beta_i U^{\alpha_i \gamma_i} \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{\alpha_i} = 1$$

Indirect utility function: Indirect utility function $v(\mathbf{p}, M)$ is given by

$$e(\mathbf{p}, v(\mathbf{p}, M)) = M$$

Compensated demand function:

$$c_i^C(\mathbf{p}, U) = \frac{\beta_i U^{\alpha_i \gamma_i} \alpha_i \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{\alpha_i - 1}}{\sum_j \beta_j U^{\alpha_j \gamma_j} \alpha_j \left[\frac{p_j}{e(\mathbf{p}, U)} \right]^{\alpha_j}}$$

Uncompensated demand function:

$$c_i^U(\mathbf{p}, M) = c_i^C(\mathbf{p}, v(\mathbf{p}, M))$$

Price elasticity of uncompensated demand:

$$\varepsilon_{ij}^C = \frac{\partial \ln c_i^C}{\partial \ln p_j} = S_j \left[1 - \alpha_i + \sum_k \alpha_k S_k - \alpha_j \right] - \delta_{ij}(1 - \alpha_i)$$

where

$$S_j \equiv \frac{p_j c_j^C}{e}$$
$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

¹For GTAP model, Hertel (1997) McDougall (2003).

Price elasticity of compensated demand:

$$\varepsilon_{ij}^U = \frac{\partial \ln c_i^U}{\partial \ln p_j} \varepsilon_{ij}^C - \eta_i S_j$$

Income elasticity of uncompensated demand:

$$\eta_i = \frac{\partial \ln c_i^U}{\partial \ln M} = \frac{1}{\sum_j \gamma_j S_j} \left[\alpha_i \gamma_i - \sum_k \alpha_k \gamma_k S_k \right] + (1 - \alpha_i) + \sum_k \alpha_k S_k$$

Case 2

Preference in Case 2 is the same as Case 1 but expression is slightly different. In Case 2, when you set $\sigma_i = \sigma \gamma_i = 1$, you get CES utility function with elasticity of substitution σ .

Expenditure function:

$$\sum_i \beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{1-\sigma_i} = 1$$

Indirect utility:

$$e(\mathbf{p}, v(\mathbf{p}, M)) = M$$

Compensated demand:

$$\begin{aligned} c_i^C(\mathbf{p}, U) &= \frac{\beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} (1 - \sigma_i) \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{-\sigma_i}}{\sum_j \beta_j^{\sigma_j} U^{(1-\sigma_j)\gamma_j} (1 - \sigma_j) \left[\frac{p_j}{e(\mathbf{p}, U)} \right]^{1-\sigma_j}} \\ &= \frac{e(\mathbf{p}, U)}{p_i} \frac{\beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} (1 - \sigma_i) \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{1-\sigma_i}}{\sum_j \beta_j^{\sigma_j} U^{(1-\sigma_j)\gamma_j} (1 - \sigma_j) \left[\frac{p_j}{e(\mathbf{p}, U)} \right]^{1-\sigma_j}} \end{aligned}$$

Uncompensated demand:

$$c_i^U(\mathbf{p}, M) = c_i^C(\mathbf{p}, v(\mathbf{p}, M))$$

Price elasticity of compensated demand:

$$\varepsilon_{ij}^C = \frac{\partial \ln c_i^C}{\partial \ln p_j} = S_j \left[\sigma_i + \sum_k (1 - \sigma_k) S_k - (1 - \sigma_j) \right] - \delta_{ij} \sigma_i$$

where

$$\begin{aligned} S_j &\equiv \frac{p_j c_j^C}{e} \\ \delta_{ij} &\equiv \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \end{aligned}$$

Price elasticity of uncompensated demand:

$$\varepsilon_{ij}^U = \frac{\partial \ln c_i^U}{\partial \ln p_j} \varepsilon_{ij}^C - \eta_i S_j$$

Income elasticity of uncompensated demand:

$$\eta_i = \frac{\partial \ln c_i^U}{\partial \ln M} = \frac{1}{\sum_j \gamma_j S_j} \left[(1 - \sigma_i) \gamma_i - \sum_k (1 - \sigma_k) \gamma_k S_k \right] + \sigma_i + \sum_k (1 - \sigma_k) S_k$$

Calibration of Paramters

Calibration of parameters in expression of Case 1.

Approach in the GTAP model

1. $\bar{c}_i^C, \bar{U}, \bar{e}, \bar{p}_i$ are determined from the benchmark data.
2. α_i and γ_i need to be determined by yourself.
3. Construct the following systems of equations:

$$c_i^C = \frac{\beta_i U^{\alpha_i \gamma_i} \alpha_i \left[\frac{p_i}{e(p, U)} \right]^{\alpha_i - 1}}{\sum_j \beta_j U^{\alpha_j \gamma_j} \alpha_j \left[\frac{p_j}{e(p, U)} \right]^{\alpha_j}} \quad i = 1, \dots, I \quad (1)$$

$$\sum_i \beta_i U^{\alpha_i \gamma_i} \left[\frac{p_i}{e} \right]^{\alpha_i} = 1 \quad (2)$$

4. Because one equation in the above $I + 1$ -equations system is redundant, drop one equation and solve I equations for β_j ($j = 1, \dots, I$).

References

- Hertel, Thomas W. ed. (1997) *Global Trade Analysis: Modeling and Applications*, New York: Cambridge University Press.
- Hertel, Thomas W., J. Mark Horridge, and Ken R. Pearson (1992) "Mending The Family Tree: A Reconciliation of The Linearization and Levels Schools of AGE Modelling," *Economic Modelling*, Vol. 9, No. 4, pp. 385-407, October.
- McDougall, Robert (2003) "A New Regional Household Demand System for GTAP." GTAP Technical Paper No. 20, September.