A Note on CDE function

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Abstract

A small note on CDE function (constant difference of elasticities function, Hertel, Horridge and Pearson 1992) used in the GTAP standard model.[1]

Case 1

An expression used in Hertel et al. (1992)Hertel (1997) McDougall (2003).

Expenditure function: Expenditure function for CDE utility function is defined implicitly as

$$\sum_i \beta_i U^{\alpha_i \gamma_i} \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{\alpha_i} = 1$$

Indirect utility function: Indirect utility function $v(\mathbf{p}, M)$ is given by

$$e(\mathbf{p}, v(\mathbf{p}, M)) = M$$

Compensated demand function:

$$c_i^C(\mathbf{p}, U) = \frac{\beta_i U^{\alpha_i \gamma_i} \alpha_i \left[\frac{p_i}{e(p,U)} \right]^{\alpha_i - 1}}{\sum_j \beta_j U^{\alpha_j \gamma_j} \alpha_j \left[\frac{p_j}{e(p,U)} \right]^{\alpha_j}}$$

Uncompensated demand function:

$$c_i^U(\mathbf{p}, M) = c_i^C(\mathbf{p}, v(\mathbf{p}, M))$$

Price elasticity of uncompensated demand:

$$\varepsilon_{ij}^C = \frac{\partial \ln c_i^C}{\partial \ln p_j} = S_j \left[1 - \alpha_i + \sum_k \alpha_k S_k - \alpha_j \right] - \delta_{ij}(1 - \alpha_i)$$

where

$$S_j \equiv \frac{p_j c_j^C}{e}$$
$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

[1]For GTAP model, Hertel (1997) McDougall (2003).

Price elasticity of compensated demand:

$$\varepsilon_{ij}^{U} = \frac{\partial \ln c_i^U}{\partial \ln p_j} \varepsilon_{ij}^{C} - \eta_i S_j$$

Income elasticity of uncompensated demand:

$$\eta_i = \frac{\partial \ln c_i^U}{\partial \ln M} = \frac{1}{\sum_j \gamma_j S_j} \left[\alpha_i \gamma_i - \sum_k \alpha_k \gamma_k S_k \right] + (1 - \alpha_i) + \sum_k \alpha_k S_k$$

Case 2

Preference in Case 2 is the same as Case 1 but expression is sligtly different. In Case 2, when you set $\sigma_i = \sigma \gamma_i = 1$, you get CES utility function with elasticity of substitution σ.

Expenditure function:

$$\sum_i \beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} \left[\frac{p_i}{e(\mathbf{p}, U)} \right]^{1-\sigma_i} = 1$$

Indirect utility:

$$e(\mathbf{p}, v(\mathbf{p}, M)) = M$$

Compensated demand:

$$\begin{aligned} c_i^C(\mathbf{p}, U) &= \frac{\beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} (1 - \sigma_i) \left[\frac{p_i}{e(p,U)} \right]^{-\sigma_i}}{\sum_j \beta_j^{\sigma_j} U^{(1-\sigma_j)\gamma_j} (1 - \sigma_j) \left[\frac{p_j}{e(p,U)} \right]^{1-\sigma_j}} \\ &= \frac{e(p,U)}{p_i} \frac{\beta_i^{\sigma_i} U^{(1-\sigma_i)\gamma_i} (1 - \sigma_i) \left[\frac{p_i}{e(p,U)} \right]^{1-\sigma_i}}{\sum_j \beta_j^{\sigma_j} U^{(1-\sigma_j)\gamma_j} (1 - \sigma_j) \left[\frac{p_j}{e(p,U)} \right]^{1-\sigma_j}} \end{aligned}$$

Uncompensated demand:

$$c_i^U(\mathbf{p}, M) = c_i^C(\mathbf{p}, v(\mathbf{p}, M))$$

Price elasticity of compensated demand:

$$\varepsilon_{ij}^{C} = \frac{\partial \ln c_i^C}{\partial \ln p_j} = S_j \left[\sigma_i + \sum_k (1 - \sigma_k) S_k - (1 - \sigma_j) \right] - \delta_{ij} \sigma_i$$

where

$$S_j \equiv \frac{p_j c_j^C}{e}$$
$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Price elasticity of uncompensated demand:

$$\varepsilon_{ij}^{U} = \frac{\partial \ln c_i^U}{\partial \ln p_j} \varepsilon_{ij}^{C} - \eta_i S_j$$

Income elasticity of uncompensated demand:

$$\eta_i = \frac{\partial \ln c_i^U}{\partial \ln M} = \frac{1}{\sum_j \gamma_j S_j} \left[(1-\sigma_i)\gamma_i - \sum_k (1-\sigma_k)\gamma_k S_k \right] + \sigma_i + \sum_k (1-\sigma_k) S_k$$

Calibration of Paramters

Calibration of parameters in expression of Case 1.

Approach in the GTAP model

1. $\bar{c}_i^C$, $\bar{U}$, $\bar{e}$, $\bar{p}_i$ are determined from the benchmark data.
2. α_i and γ_i need to be determined by yourself.
3. Construct the following systems of equations:

$$c_i^C = \frac{\beta_i U^{\alpha_i \gamma_i} \alpha_i \left[\frac{p_i}{e(p,U)} \right]^{\alpha_i - 1}}{\sum_j \beta_j U^{\alpha_j \gamma_j} \alpha_j \left[\frac{p_j}{e(p,U)} \right]^{\alpha_j}} \qquad i = 1, \cdots, I \tag{1}$$

$$\sum_i \beta_i U^{\alpha_i \gamma_i} \left[\frac{p_i}{e} \right]^{\alpha_i} = 1 \tag{2}$$

4. Because one equation in the above $I+1$-equations system is redundant, drop one equation and solve I equations for β_j ($j = 1, \dots, I$).

References

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